

# Relativistic Chern–Simons–Higgs Vortex Equations

Xiaosen Han

Institute of Contemporary Mathematics

Henan University

Kaifeng, Henan 475000, PR China

Yisong Yang

Department of Mathematics

Polytechnic School of Engineering

New York University

Brooklyn, New York 11201, USA

&

NYU-ECNU Institute of Mathematical Sciences

New York University - Shanghai

3663 North Zhongshan Road, Shanghai 200062, PR China

## Abstract

An existence theorem is established for the solutions to the non-Abelian relativistic Chern–Simons–Higgs vortex equations over a doubly periodic domain when the gauge group  $G$  assumes the most general and important prototype form,  $G = SU(N)$ .

**Mathematics subject classification (2010).** 35C08, 35J50, 35Q70, 81E13

## 1 Introduction

Let  $K = (K_{ij})$  be the Cartan matrix of a semi-simple Lie algebra  $L$ . Recall that the Toda system is a system of nonlinear elliptic equations over  $\mathbb{R}^2$ , of exponential nonlinearities, of the form

$$\Delta u_i = -\lambda \sum_{j=1}^r K_{ij} e^{u_j}, \quad i = 1, \dots, r, \quad (1.1)$$

where  $r$  is the rank of  $L$ . This system is known to be integrable in general [21, 22, 31–34, 42, 47] and arises in the study of non-Abelian monopoles [22, 47, 53] and nonrelativistic Chern–Simons–Higgs vortices [13–16, 25]. Interestingly, when  $r = 1$ , that is, when the Cartan

subalgebra of  $L$  is Abelian such as when  $L$  is the Lie algebra of  $SU(2)$ , (1.1) reduces to the classical Liouville equation [38]

$$\Delta u = -\lambda e^u, \quad (1.2)$$

whose solutions may be constructed by all the integration methods known, such as separation of variables, inverse scattering, the Bäcklund transformation, etc., and is often used as an illustrative example. It is well known that the Liouville equation and its extensions arise also in differential geometry [2, 5, 29, 30] and have been the focus of various studies on their analytic aspects [3, 6–12, 35]. More recently, some extensive work on (1.1) has been carried out as well aimed at the classification of solutions [28, 36], understanding its fine analytic structures [39–41], and establishment of bubbling behavior of solutions [27, 37, 45].

On the other hand, however, when one considers relativistic Chern–Simons–Higgs vortices [24, 26, 56], the governing system of equations [14–16, 56] is

$$\Delta u_i = \lambda \sum_{j=1}^r \sum_{k=1}^r K_{kj} K_{ji} e^{u_j} e^{u_k} - \lambda \sum_{j=1}^r K_{ji} e^{u_j}, \quad i = 1, \dots, r, \quad (1.3)$$

which deviates from (1.1) significantly in that, even in the scalar case where  $r = 1$ , the system is nonintegrable [48]. Thus, it is perceivable that the analytic structure of (1.3) would be much more complicated than that of (1.1). Due to its applicability in anyon physics [18–20, 54] and challenging mathematical content, it will be desirable to develop an existence theory for the relativistic vortex equations (1.3), in which the sources terms resulting from the presence of vortices are temporarily neglected in order to facilitate our discussion.

In [55], a systematic study is conducted to establish an existence theorem for the so-called topological solutions, realizing the spontaneous symmetry breaking or the celebrated Higgs mechanism of the model, over the full plane  $\mathbb{R}^2$ , to the equations (1.3), where the Cartan matrix is of a general form. Another type of solutions of great interest are called the Abrikosov vortices [1] or vortex condensates for which the equations are to be solved over a doubly-periodic lattice domain. In field-theoretic formalism, such a structure is realized by imposing the 't Hooft [50] boundary condition on gauge and matter fields [49, 52, 56]. Surprisingly, although the system is now considered over a compact domain, double periodicity greatly complicates the problem as evidenced [46] already in the nonrelativistic Liouville equation situation where one needs to use the Weierstrass elliptic functions as building blocks for solutions. In the relativistic situation, although there is a variational principle, the action functional is not bounded from below and one has to consider a constrained minimization problem. However, the presence of the constraints leads to a Lagrange multiplier issue so that it prevents one from recovering the original equations of motion. In order to overcome this difficulty, one has to consider an inequality-constrained problem instead and bypass the Lagrange multiplier problem with achieving an interior minimum. As a consequence, the progress in developing an existence theory for the doubly periodic solutions of the relativistic Chern–Simons–Higgs vortex equations has been slow and sporadic. Specifically, in [4],

an inequality-constrained minimization method was first used to establish the existence of solutions for the scalar case of (1.3), namely  $r = 1$  or  $G = SU(2)$ , and, in [44], the method in [4] was remarkably extended and refined to tackle the first non-scalar case of (1.3), namely  $r = 2$  or  $G = SU(3)$ , in which there are two inequality constraints characterizing the solvability of two quadratic constraints. The study on the next important non-scalar case of (1.3), that is,  $r = 3$  or  $G = SU(4)$ , had been unsuccessful due to the difficulty in resolving more than two quadratic constraints simultaneously until very recently a new idea based on an implicit-function theorem argument was implemented to resolve three coupled quadratic constraints, which enables the establishment of an existence theorem for  $G = SU(4)$  in [23].

The purpose of the present paper is to establish an existence theorem for the doubly periodic solutions of (1.3) for the most general situation,  $G = SU(N)$  ( $N \geq 2$ ) or  $r = N - 1 \geq 1$ , adapting the implicit function method initiated in [23] in handling multiple quadratic constraints. Instead of a ‘squeeze-to-the-middle’ approach in [23] for resolving the constraints, however, we use here an ordered iterative scheme which is effective and easier to implement in the general situation. In the next section, we state our main existence theorem. In the section after, we prove the theorem. In the last section we end the paper with some remarks.

*Note added upon acceptance.* This work grew out of an earlier version of [23] in which only the  $SU(4)$  problem was resolved. Since the submission of the present work, further development has been achieved to resolve the general situation when  $G$  is a simple Lie group by using a degree-theory method formalism, which greatly expands [23] into its current updated version seen.

## 2 Existence theorem

With the source terms in the presence of multiply distributed vortices, the non-Abelian relativistic Chern–Simons–Higgs equations are [14–16, 56]

$$\Delta u_i = \lambda \left( \sum_{j=1}^r \sum_{k=1}^r K_{kj} K_{ji} e^{u_j} e^{u_k} - \sum_{j=1}^r K_{ji} e^{u_j} \right) + 4\pi \sum_{j=1}^{N_i} \delta_{p_{ij}}(x), \quad i = 1, \dots, r, \quad (2.1)$$

where  $\delta_p$  denotes the Dirac measure concentrated at the point  $p$ ,  $\lambda > 0$  is a coupling constant, and the equations are considered over a doubly periodic domain  $\Omega$  resembling a lattice cell housing a distribution of dually charged vortices located at  $p_{ij}, j = 1, \dots, N_i, i = 1, \dots, r$ .

We focus on the system (2.1) with  $G = SU(n + 1), n \geq 2$ . The Cartan matrix  $K$  for  $SU(n + 1)$  is an  $n \times n$  matrix given by

$$K \equiv \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}. \quad (2.2)$$

It is easy to check that  $K^{-1}$  is symmetric with entries given by

$$(K^{-1})_{ij} = \frac{i(n+1-j)}{n+1}, \quad i \leq j, \quad i = 1, \dots, n. \quad (2.3)$$

Then after the translation

$$u_i \rightarrow u_i + \ln R_i \quad \text{with} \quad R_i \equiv \sum_{j=1}^n (K^{-1})_{ij} = \frac{i(n+1-i)}{2}, \quad i = 1, \dots, n, \quad (2.4)$$

the system (2.1) can be rewritten as

$$\Delta u_i = \lambda \left( \sum_{j=1}^n \sum_{k=1}^n \tilde{K}_{jk} \tilde{K}_{ij} e^{u_j} e^{u_k} - \sum_{j=1}^n \tilde{K}_{ij} e^{u_j} \right) + 4\pi \sum_{j=1}^{N_i} \delta_{p_{ij}}(x), \quad i = 1, \dots, n, \quad (2.5)$$

or in a vector form,

$$\Delta \mathbf{u} = \lambda \tilde{K} \mathbf{U} \tilde{K} (\mathbf{U} - \mathbf{1}) + 4\pi \mathbf{s}, \quad (2.6)$$

where

$$\tilde{K} \equiv KR = \begin{pmatrix} n & -(n-1) & 0 & \cdots & \cdots & 0 \\ -\frac{n}{2} & 2(n-1) & -\frac{3(n-2)}{2} & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \cdots & \cdots & -\frac{(i-1)(n+2-i)}{2} & i(n+1-i) & -\frac{(i+1)(n-i)}{2} & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & -\frac{3(n-2)}{2} & 2(n-1) & -\frac{n}{2} \\ 0 & \cdots & \cdots & 0 & -(n-1) & n \end{pmatrix}, \quad (2.7)$$

$$R \equiv \text{diag} \{R_1, \dots, R_n\}, \quad \mathbf{1} = (1, \dots, 1)^\tau, \quad (2.8)$$

$$\mathbf{u} = (u_1, \dots, u_n)^\tau, \quad \mathbf{U} = \text{diag} \{e^{u_1}, \dots, e^{u_n}\}, \quad \mathbf{U} = (e^{u_1}, \dots, e^{u_n})^\tau, \quad (2.9)$$

$$\mathbf{s} = \left( \sum_{s=1}^{N_1} \delta_{p_{1s}}, \dots, \sum_{s=1}^{N_n} \delta_{p_{ns}} \right)^\tau. \quad (2.10)$$

By the definition of  $\tilde{K}$  given in (2.7), we obtain the following simple facts

$$\tilde{K}^{-1} = R^{-1} K^{-1}, \quad \tilde{K}^{-1} \mathbf{1} = \mathbf{1}, \quad K^{-1} \mathbf{1} = R \mathbf{1}, \quad (2.11)$$

which will be repeatedly used later in this paper.

We are interested in the existence of solutions of (2.5) or (2.6) over a doubly periodic domain  $\Omega$ . Our main result reads as follows.

**Theorem 2.1** *Consider the nonlinear elliptic system (2.5) or (2.6) over a doubly periodic domain  $\Omega$  in  $\mathbb{R}^2$ . For any given points  $p_{i1}, \dots, p_{iN_i} \in \Omega$  ( $i = 1, \dots, n$ ), which need not to be distinct, the following conclusions hold.*

(i) *(Necessary condition for existence) If*

$$\lambda \leq \lambda_0 \equiv \frac{16\pi \sum_{i=1}^n \sum_{j=1}^n (K^{-1})_{ij} N_j}{|\Omega| \sum_{i=1}^n \sum_{j=1}^n (K^{-1})_{ij}}, \quad (2.12)$$

*there is no solution to the system, where the entries of  $K^{-1}$  are given by (2.3). In other words, a solution can exist only when  $\lambda$  is larger than the right-hand side of (2.12).*

(ii) *(Sufficient condition for existence) There exists some  $\lambda_1 > \lambda_0$  such that when  $\lambda > \lambda_1$  the system admits a solution over  $\Omega$ .*

(iii) *(Asymptotic behavior) The solution  $(u_1, \dots, u_n)$  obtained above satisfies*

$$\lim_{\lambda \rightarrow \infty} \int_{\Omega} (e^{u_i} - 1)^2 dx = 0, \quad i = 1, \dots, n. \quad (2.13)$$

(iv) *(Quantized integrals) If  $(u_1, \dots, u_n)$  is a solution then there hold the quantized integrals*

$$\int_{\Omega} \left( \sum_{j=1}^n \sum_{k=1}^n \tilde{K}_{jk} \tilde{K}_{ij} e^{u_j} e^{u_k} - \sum_{j=1}^n \tilde{K}_{ij} e^{u_j} \right) dx = -\frac{4\pi N_i}{\lambda}, \quad i = 1, \dots, n. \quad (2.14)$$

Note that when the vortex numbers  $N_i$  ( $i = 1, \dots, n$ ) are the same, say  $m$ , our necessary condition (2.12) reduces to that for the  $U(1)$  case as in [4] as follows

$$\lambda \leq \frac{16\pi m}{|\Omega|}. \quad (2.15)$$

which is a comfort and also surprising since we are now considering a non-Abelian and non-scalar situation.

### 3 Proof of theorem

In this section we apply a constrained minimization procedure developed in [4], which was later modified in [44], to establish the existence of doubly periodic solutions to (2.5) (2.6) when  $n = 2$ . We carry out our proof in several steps. First we show that the condition (2.12) implies nonexistence of solutions as stated and we explore a variational structure of our equations. Next we show how to resolve multiple constraints using an iterative scheme and an implicit function argument. We then conduct a constrained minimization procedure and show that there is a solution to the minimization problem. In the subsequent subsection, we establish some suitable estimates which ensures that the minimum point obtained must

be an interior minimum, thus ruling out the Lagrange multiplier issue. In the last two subsections, we show that the interior minimum point obtained is a classical solution of the original vortex equations and we then establish the stated asymptotic behavior of solutions as  $\lambda \rightarrow \infty$  and quantized integrals.

### 3.1 Necessary condition and variational structure

Let  $u_i^0$  be the solution of the following problem (see [2])

$$\Delta u_i^0 = 4\pi \sum_{s=1}^{N_i} \delta_{p_{is}} - \frac{4\pi N_i}{|\Omega|}, \quad \int_{\Omega} u_i^0 dx = 0, \quad i = 1, \dots, n, \quad (3.1)$$

and  $u_i = u_i^0 + v_i$ ,  $i = 1, \dots, n$ . Let us introduce the notation ( $n$ -vectors)

$$\mathbf{v} = (v_1, \dots, v_n)^\tau, \quad \mathbf{N} = (N_1, \dots, N_n)^\tau, \quad \mathbf{0} = (0, \dots, 0)^\tau. \quad (3.2)$$

Then we reformulate the system (2.5) or (2.6) as

$$\Delta v_i = \lambda \left( \sum_{j=1}^n \sum_{k=1}^n \tilde{K}_{jk} \tilde{K}_{ij} e^{u_j^0 + v_j} e^{u_k^0 + v_k} - \sum_{j=1}^n \tilde{K}_{ij} e^{u_j^0 + v_j} \right) + \frac{4\pi N_i}{|\Omega|}, \quad i = 1, \dots, n, \quad (3.3)$$

or

$$\Delta \mathbf{v} = \lambda \tilde{K} \mathbf{U} \tilde{K} (\mathbf{U} - \mathbf{1}) + \frac{4\pi}{|\Omega|} \mathbf{N}, \quad (3.4)$$

with the understanding that

$$\mathbf{U} = \text{diag} \left\{ e^{u_1^0 + v_1}, \dots, e^{u_n^0 + v_n} \right\}, \quad \mathbf{U} = (e^{u_1^0 + v_1}, \dots, e^{u_n^0 + v_n})^\tau. \quad (3.5)$$

We first present a necessary condition for the existence of solution to (3.3) or (3.4). For any solution  $\mathbf{v}$  of (3.4), taking integration over  $\Omega$ , we obtain the natural constraint

$$\int_{\Omega} \tilde{K} \mathbf{U} \tilde{K} (\mathbf{U} - \mathbf{1}) dx + \frac{4\pi}{\lambda} \mathbf{N} = \mathbf{0}. \quad (3.6)$$

Multiplying both sides of (3.6) by  $K^{-1}$ , we have

$$\int_{\Omega} R \mathbf{U} K R (\mathbf{U} - \mathbf{1}) dx + \frac{4\pi}{\lambda} K^{-1} \mathbf{N} = \mathbf{0}. \quad (3.7)$$

Then noting (3.7), the positive definiteness of  $K$ , and the fact that  $K^{-1} \mathbf{1} = R \mathbf{1}$ , we have

$$\begin{aligned} 0 &= \int_{\Omega} (R \mathbf{U})^\tau K R (\mathbf{U} - \mathbf{1}) dx + \frac{4\pi}{\lambda} \mathbf{1}^\tau K^{-1} \mathbf{N} \\ &= \int_{\Omega} \left( R \left[ \mathbf{U} - \frac{1}{2} \mathbf{1} \right] \right)^\tau K \left( R \left[ \mathbf{U} - \frac{1}{2} \mathbf{1} \right] \right) dx - \frac{|\Omega|}{4} (R \mathbf{1})^\tau K (R \mathbf{1}) + \frac{4\pi}{\lambda} \mathbf{1}^\tau K^{-1} \mathbf{N} \\ &> -\frac{|\Omega|}{4} \mathbf{1}^\tau K^{-1} \mathbf{1} + \frac{4\pi}{\lambda} \mathbf{1}^\tau K^{-1} \mathbf{N}, \end{aligned} \quad (3.8)$$

which implies

$$\lambda > \frac{16\pi \mathbf{1}^\tau K^{-1} \mathbf{N}}{|\Omega| \mathbf{1}^\tau K^{-1} \mathbf{1}}. \quad (3.9)$$

That is, (3.9) spells out a necessary condition for the existence of solutions to (3.4). Hence the first conclusion of Theorem 2.1 follows.

Now we aim to find a variational principle for the equations (3.4). After a simple computation, we see that the matrix  $\tilde{K}$  admits a decomposition

$$\tilde{K} = \tilde{P} \tilde{S}, \quad (3.10)$$

with

$$\tilde{P} \equiv \text{diag} \left\{ \tilde{P}_1, \dots, \tilde{P}_n \right\}, \quad \tilde{P}_i \equiv \frac{n}{i(n+1-i)}, \quad i = 1, \dots, n, \quad (3.11)$$

$$\tilde{S} \equiv \begin{pmatrix} n & -(n-1) & 0 & \dots & 0 \\ -(n-1) & \frac{4(n-1)^2}{n} & -\frac{3(n-1)(n-2)}{n} & \dots & \dots \\ \ddots & \ddots & \ddots & \vdots & \vdots \\ \dots & -\tilde{S}_{ii-1} & \tilde{S}_{ii} & -\tilde{S}_{ii+1} & \dots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \dots & \dots & -\frac{3(n-1)(n-2)}{n} & \frac{4(n-1)^2}{n} & -(n-1) \\ 0 & \dots & 0 & -(n-1) & n \end{pmatrix}, \quad (3.12)$$

which is a tridiagonal matrix with

$$\begin{cases} \tilde{S}_{ii-1} & \equiv \frac{(i-1)i(n+2-i)(n+1-i)}{2n}, \\ \tilde{S}_{ii} & \equiv \frac{i^2(n+1-i)^2}{2n}, \\ \tilde{S}_{ii+1} & \equiv \frac{i(i+1)(n+1-i)(n-i)}{2n}, \end{cases} \quad i = 3, \dots, n-2. \quad (3.13)$$

Then we may rewrite (3.4) equivalently as

$$\Delta M \mathbf{v} = \lambda U \tilde{S} (\mathbf{U} - \mathbf{1}) + \frac{\mathbf{b}}{|\Omega|}, \quad (3.14)$$

where

$$M \equiv \tilde{P}^{-1} \tilde{K}^{-1} = \frac{2}{n} K^{-1}, \quad \tilde{P}^{-1} = \text{diag} \left\{ \tilde{P}_1^{-1}, \dots, \tilde{P}_n^{-1} \right\}, \quad (3.15)$$

$$\tilde{P}_i^{-1} = \frac{i(n+1-i)}{n}, \quad i = 1, \dots, n, \quad (3.16)$$

and

$$\mathbf{b} = (b_1, \dots, b_n)^\tau \equiv 4\pi M \mathbf{N}. \quad (3.17)$$

By the definition (3.17) for  $\mathbf{b}$ , we easily find that  $b_i > 0$ ,  $i = 1, \dots, n$ .

We use  $W^{1,2}(\Omega)$  to denote the Sobolev space of scalar-valued or vector-valued  $\Omega$ -periodic  $L^2$  functions with their derivatives also belonging to  $L^2(\Omega)$ .

It may be examined that the equations (3.14) are the Euler–Larange equations of the action functional

$$I(\mathbf{v}) = \frac{1}{2} \sum_{i=1}^2 \int_{\Omega} \partial_i \mathbf{v}^{\tau} M \partial_i \mathbf{v} dx + \frac{\lambda}{2} \int_{\Omega} (\mathbf{U} - \mathbf{1})^{\tau} \tilde{S}(\mathbf{U} - \mathbf{1}) dx + \frac{1}{|\Omega|} \int_{\Omega} \mathbf{b}^{\tau} \mathbf{v} dx, \quad (3.18)$$

where we use the notation (3.12), (3.15) and (3.17) throughout this paper.

In the following subsections we will use a constrained minimization approach to find the critical points of the functional  $I$ .

### 3.2 Multiple constraints

To start our constrained minimization process, we need to find some suitable constraints subject to which the functional  $I$  will be minimized.

Note the space  $W^{1,2}(\Omega)$  can be decomposed as

$$W^{1,2}(\Omega) = \mathbb{R} \oplus \dot{W}^{1,2}(\Omega), \quad (3.19)$$

where

$$\dot{W}^{1,2}(\Omega) \equiv \left\{ w \in W^{1,2}(\Omega) \left| \int_{\Omega} w dx = 0 \right. \right\}, \quad (3.20)$$

is a closed subspace of  $W^{1,2}(\Omega)$ .

Then, for  $v_i \in W^{1,2}(\Omega)$ , we have the decomposition

$$v_i = c_i + w_i, \quad c_i \in \mathbb{R}, \quad w_i \in \dot{W}^{1,2}(\Omega), \quad i = 1, \dots, n. \quad (3.21)$$

If  $\mathbf{v} \in W^{1,2}(\Omega)$  satisfies the constraint (3.6), which is equivalent to

$$\int_{\Omega} \mathbf{U} \tilde{S}(\mathbf{U} - \mathbf{1}) dx + \frac{\mathbf{b}}{\lambda} = \mathbf{0}, \quad (3.22)$$

then by the decomposition (3.21) with  $\mathbf{w} = (w_1, \dots, w_n)^{\tau}$ , we obtain

$$e^{2c_1} a_{11} - e^{c_1} P_1(\mathbf{w}; e^{c_2}) + \frac{b_1}{n\lambda} = 0, \quad (3.23)$$

$$e^{2c_i} a_{ii} - e^{c_i} P_i(\mathbf{w}; e^{c_{i-1}}, e^{c_{i+1}}) + \frac{nb_i}{i^2(n+1-i)^2\lambda} = 0, \quad i = 2, \dots, n-1, \quad (3.24)$$

$$e^{2c_n} a_{nn} - e^{c_n} P_n(\mathbf{w}; e^{c_{n-1}}) + \frac{b_n}{n\lambda} = 0, \quad (3.25)$$



where and in the sequel the notation

$$P_1(\mathbf{w}; e^{c_2}) \equiv \frac{a_1}{n} + \frac{(n-1)a_{12}}{n} e^{c_2}, \quad (3.26)$$

$$P_i(\mathbf{w}; e^{c_{i-1}}, e^{c_{i+1}}) \equiv \frac{a_i}{i(n+1-i)} + \frac{(i-1)(n+2-i)a_{ii-1}}{2i(n+1-i)} e^{c_{i-1}} \\ + \frac{(i+1)(n-i)a_{ii+1}}{2i(n+1-i)} e^{c_{i+1}}, \quad i = 2, \dots, n-1, \quad (3.27)$$

$$P_n(\mathbf{w}; e^{c_{n-1}}) \equiv \frac{a_n}{n} + \frac{(n-1)a_{nn-1}}{n} e^{c_{n-1}}, \quad (3.28)$$

and the definition

$$a_i \equiv a_i(w_i) \equiv \int_{\Omega} e^{u_i^0 + w_i} dx, \quad (3.29)$$

$$a_{ij} \equiv a_{ij}(w_i, w_j) \equiv \int_{\Omega} e^{u_i^0 + u_j^0 + w_i + w_j} dx, \quad i, j = 1, \dots, n, \quad (3.30)$$

are used.

Then, we see that, for any  $\mathbf{w} \in \dot{W}^{1,2}(\Omega)$ , the equations (3.23)–(3.25) are solvable in

$$\mathbf{c} \equiv (c_1, \dots, c_n)^\tau, \quad (3.31)$$

only if

$$P_1^2(\mathbf{w}; e^{c_2}) \geq \frac{4b_1 a_{11}}{n\lambda}, \quad (3.32)$$

$$P_i^2(\mathbf{w}; e^{c_{i-1}}, e^{c_{i+1}}) \geq \frac{4nb_i a_{ii}}{i^2(n+1-i)^2\lambda}, \quad i = 2, \dots, n-1, \quad (3.33)$$

$$P_n^2(\mathbf{w}; e^{c_{n-1}}) \geq \frac{4b_n a_{nn}}{n\lambda}, \quad (3.34)$$

are satisfied which can be ensured by the following simpler inequality-type constraints

$$a_i^2 \geq \frac{4nb_i a_{ii}}{\lambda}, \quad i = 1, \dots, n, \quad (3.35)$$

or expressed explicitly as

$$\left( \int_{\Omega} e^{u_i^0 + w_i} dx \right)^2 \geq \frac{4nb_i}{\lambda} \int_{\Omega} e^{2u_i^0 + 2w_i} dx, \quad i = 1, \dots, n. \quad (3.36)$$

Now we define the admissible set

$$\mathcal{A} \equiv \left\{ \mathbf{w} \mid \mathbf{w} \in \dot{W}^{1,2}(\Omega) \text{ satisfies (3.36)} \right\}. \quad (3.37)$$

Thus, for any  $\mathbf{w} \in \mathcal{A}$ , we can find a solution of the equations (3.23)–(3.25) with respect to  $(c_1, \dots, c_n)$  by solving the following coupled equations

$$\begin{aligned} e^{c_1} &= \frac{P_1(\mathbf{w}; e^{c_2}) + \sqrt{P_1^2(\mathbf{w}; e^{c_2}) - \frac{4b_1a_{11}}{n\lambda}}}{2a_{11}} \\ &\equiv f_1(e^{c_2}), \end{aligned} \quad (3.38)$$

$$\begin{aligned} e^{c_i} &= \frac{P_i(\mathbf{w}; e^{c_{i-1}}, e^{c_{i+1}}) + \sqrt{P_i^2(\mathbf{w}; e^{c_{i-1}}, e^{c_{i+1}}) - \frac{4nb_ia_{ii}}{i^2(n+1-i)^2\lambda}}}{2a_{ii}} \\ &\equiv f_i(e^{c_{i-1}}, e^{c_{i+1}}), \quad i = 2, \dots, n-1, \end{aligned} \quad (3.39)$$

$$\begin{aligned} e^{c_n} &= \frac{P_n(\mathbf{w}; e^{c_{n-1}}) + \sqrt{P_n^2(\mathbf{w}; e^{c_{n-1}}) - \frac{4b_na_{nn}}{n\lambda}}}{2a_{nn}} \\ &\equiv f_n(e^{c_{n-1}}). \end{aligned} \quad (3.40)$$

For any  $\mathbf{w} \in \mathcal{A}$ , to solve the system of the equations (3.38)–(3.40) with respect to  $(c_1, \dots, c_n)$ , we will convert the system into a single equation.

For  $n \geq 2$ , we need to find a solution of the system

$$t_1 - f_1(t_2) = 0, \quad (3.41)$$

$$t_i - f_i(t_{i-1}, t_{i+1}) = 0, \quad i = 2, \dots, n-1, \quad (3.42)$$

$$t_n - f_n(t_{n-1}) = 0. \quad (3.43)$$

**Proposition 3.1** *For any  $\mathbf{w} \in \mathcal{A}$  and  $n \geq 2$ , the system (3.41)–(3.43) admits a unique solution in  $(0, \infty)^n$ .*

**Proof.** When  $n = 2, 3$ , the system can be transformed into a single equation directly, which was solved in [44] and [23], respectively. However, for  $n \geq 4$ , it difficult to reduce the system (3.41)–(3.43) into a single equation directly. Here we use the implicit function theorem to overcome this difficulty.

For  $n \geq 4$ , using the first two equations of the system (3.41)–(3.43), we have the relation

$$F_2(t_2, t_3) \equiv t_2 - f_2(f_1(t_2), t_3) = 0, \quad t_2, t_3 > 0. \quad (3.44)$$

We first show that the relation (3.44) may uniquely determine an implicit function

$$t_2 = g_2(t_3) > 0, \quad t_3 > 0. \quad (3.45)$$

By the expression (3.38)–(3.40), we have

$$\begin{aligned} \frac{df_1(t_2)}{dt_2} &= \frac{1}{2a_{11}} \left\{ \frac{(n-1)a_{12}}{n} + \frac{\frac{(n-1)a_{12}}{n}P_1(\mathbf{w}; t_2)}{\sqrt{P_1^2(\mathbf{w}; t_2) - \frac{4b_1a_{11}}{n\lambda}}} \right\} \\ &= \frac{\frac{(n-1)a_{12}}{n}f_1(t_2)}{\sqrt{P_1^2(\mathbf{w}; t_2) - \frac{4b_1a_{11}}{n\lambda}}}. \end{aligned} \quad (3.46)$$

Similarly, we obtain

$$\frac{\partial f_2(t_1, t_3)}{\partial t_1} = \frac{\frac{na_{21}}{4(n-1)}f_2(t_1, t_3)}{\sqrt{P_2^2(\mathbf{w}; t_1, t_3) - \frac{nb_2a_{22}}{(n-1)^2\lambda}}}. \quad (3.47)$$

Then, using (3.46)–(3.47) and the constraints (3.35), we have

$$\begin{aligned} \frac{\partial F_2(t_2, t_3)}{\partial t_2} &= 1 - \frac{\partial f_2(t_1, t_3)}{\partial t_1} \Big|_{t_1=f_1(t_2)} \frac{df_1(t_2)}{dt_2} \\ &= 1 - \frac{\frac{1}{4}a_{12}^2 f_1(t_2) f_2(f_1(t_2), t_3)}{\sqrt{P_1^2(\mathbf{w}; t_2) - \frac{4b_1a_{11}}{n\lambda}} \sqrt{P_2^2(\mathbf{w}; f_1(t_2), t_3) - \frac{nb_2a_{22}}{(n-1)^2\lambda}}} \\ &> 1 - \frac{\frac{1}{4}a_{12}^2 f_1(t_2) f_2(f_1(t_2), t_3)}{\frac{(n-1)a_{12}}{n} t_2 \frac{na_{12}}{4(n-1)} f_1(t_2)} \\ &= 1 - \frac{f_2(f_1(t_2), t_3)}{t_2} = \frac{F_2(t_2, t_3)}{t_2} = 0, \quad t_2, t_3 > 0. \end{aligned} \quad (3.48)$$

Therefore, from (3.48) and the implicit function theorem we see that there exists a unique implicit function

$$t_2 = g_2(t_3) > 0, \quad t_3 > 0, \quad \text{such that } F_2(g_2(t_3), t_3) \equiv 0, \quad t_3 > 0. \quad (3.49)$$

Similarly, combining (3.49) and the third ( $i = 3$ ) equation of the system (3.41)–(3.43) gives the relation

$$F_3(t_3, t_4) \equiv t_3 - f_3(g_2(t_3), t_4) = 0, \quad t_3, t_4 > 0, \quad (3.50)$$

which also determines a unique implicit function

$$t_3 = g_3(t_4) > 0, \quad t_4 > 0, \quad (3.51)$$

such that

$$F_3(g_3(t_4), t_4) \equiv 0, \quad t_4 > 0. \quad (3.52)$$

Repeating the above procedure, we obtain that there exists a family of uniquely determined implicit functions

$$t_i = g_i(t_{i+1}) > 0, \quad t_{i+1} > 0, \quad i = 4, \dots, n-1, \quad (3.53)$$

satisfying

$$F_i(t_i, t_{i+1}) \equiv t_i - f_i(g_{i-1}(t_i), t_{i+1}) = 0, \quad t_i, t_{i+1} > 0, \quad i = 4, \dots, n-1. \quad (3.54)$$

Therefore, in view of (3.49), (3.51)–(3.54), to solve the system (3.41)–(3.43), it is equivalent to solve the following single equation

$$F(t_n) \equiv t_n - f_n(g_{n-1}(t_n)) = 0, \quad t_n \geq 0. \quad (3.55)$$

In other words, to prove our proposition, we just need to show that  $F(\cdot)$  admits a unique positive zero.

We easily see that

$$F(0) < 0. \quad (3.56)$$

Next we prove that

$$\lim_{t_n \rightarrow \infty} F(t_n) = \infty. \quad (3.57)$$

Noting (3.41)–(3.43), after a direct computation, we obtain

$$\lim_{t_n \rightarrow \infty} \frac{t_1}{t_n} = \frac{(n-1)a_{12}}{na_{11}} \lim_{t_n \rightarrow \infty} \frac{t_2}{t_n}, \quad (3.58)$$

$$\begin{aligned} \lim_{t_n \rightarrow \infty} \frac{t_i}{t_n} &= \frac{1}{a_{ii}} \left\{ \frac{(i-1)(n+2-i)a_{ii-1}}{2i(n+1-i)} \lim_{t_n \rightarrow \infty} \frac{t_{i-1}}{t_n} \right. \\ &\quad \left. + \frac{(i+1)(n-i)a_{ii+1}}{2i(n+1-i)} \lim_{t_n \rightarrow \infty} \frac{t_{i+1}}{t_n} \right\}, \quad i = 2, \dots, n-2, \end{aligned} \quad (3.59)$$

$$\lim_{t_n \rightarrow \infty} \frac{t_{n-1}}{t_n} = \frac{1}{a_{n-1n-1}} \left\{ \frac{3(n-2)a_{n-1n-2}}{4(n-1)} \lim_{t_n \rightarrow \infty} \frac{t_{n-2}}{t_n} + \frac{na_{n-1n}}{4(n-1)} \right\}. \quad (3.60)$$

Then we infer from (3.58)–(3.60) and Hölder's inequality that

$$\lim_{t_n \rightarrow \infty} \frac{t_i}{t_n} \leq \frac{(n-i)a_{ii+1}}{(n-i+1)a_{ii}} \lim_{t_n \rightarrow \infty} \frac{t_{i+1}}{t_n}, \quad i = 2, \dots, n-1. \quad (3.61)$$

In particular, we have

$$\lim_{t_n \rightarrow \infty} \frac{g_{n-1}(t_n)}{t_n} = \lim_{t_n \rightarrow \infty} \frac{t_{n-1}}{t_n} \leq \frac{a_{n-1n}}{2a_{n-1n-1}}. \quad (3.62)$$

Then using (3.62) and Hölder's inequality again, we have

$$\begin{aligned} \lim_{t_n \rightarrow \infty} \frac{F(t_n)}{t_n} &= 1 - \lim_{t_n \rightarrow \infty} \frac{f_n(g_{n-1}(t_n))}{t_n} \\ &= 1 - \frac{(n-1)a_{nn-1}}{na_{nn}} \lim_{t_n \rightarrow \infty} \frac{g_{n-1}(t_n)}{t_n} \\ &\geq 1 - \frac{(n-1)a_{nn-1}^2}{2na_{n-1n-1}a_{nn}} \\ &\geq 1 - \frac{n-1}{2n} \\ &= \frac{n+1}{2n}. \end{aligned} \quad (3.63)$$

Hence (3.63) implies the desired limit (3.57).

Consequently, from (3.56)–(3.57), we conclude that  $F(\cdot)$  admits at least one positive zero. In the following, we prove the uniqueness of the zero.

Using (3.38)–(3.40) and the constraints (3.35) and after a direct computation, we have

$$\begin{aligned}\frac{\partial f_{n-1}(t_{n-2}, t_n)}{\partial t_n} &= \frac{\frac{na_{n-1n}}{4(n-1)} f_{n-1}(t_{n-2}, t_n)}{\sqrt{P_{n-1}^2(\mathbf{w}; t_{n-2}, t_n) - \frac{nb_{n-1}a_{n-1n-1}}{(n-1)^2\lambda}}} \\ &\leq \frac{\frac{na_{n-1n}}{4(n-1)} f_{n-1}(t_{n-2}, t_n)}{\frac{3(n-2)a_{n-1n-2}}{4(n-1)} t_{n-2} + \frac{na_{n-1n}}{4(n-1)} t_n},\end{aligned}\quad (3.64)$$

$$\begin{aligned}\frac{\partial f_{n-1}(t_{n-2}, t_n)}{\partial t_{n-2}} &= \frac{\frac{3(n-2)a_{n-1n-2}}{4(n-1)} f_{n-1}(t_{n-2}, t_n)}{\sqrt{P_{n-1}^2(\mathbf{w}; t_{n-2}, t_n) - \frac{nb_{n-1}a_{n-1n-1}}{(n-1)^2\lambda}}} \\ &\leq \frac{\frac{3(n-2)a_{n-1n-2}}{4(n-1)} f_{n-1}(t_{n-2}, t_n)}{\frac{3(n-2)a_{n-1n-2}}{4(n-1)} t_{n-2} + \frac{na_{n-1n}}{4(n-1)} t_n},\end{aligned}\quad (3.65)$$

$$\begin{aligned}\frac{\partial f_{n-2}(t_{n-3}, t_{n-1})}{\partial t_{n-1}} &= \frac{\frac{(n-1)a_{n-2n-1}}{3(n-2)} f_{n-2}(t_{n-3}, t_{n-1})}{\sqrt{P_{n-2}^2(\mathbf{w}; t_{n-3}, t_{n-1}) - \frac{4nb_{n-2}a_{n-2n-2}}{9(n-2)^2\lambda}}} \\ &< \frac{\frac{(n-1)a_{n-2n-1}}{3(n-2)} f_{n-2}(t_{n-3}, t_{n-1})}{\frac{(n-1)a_{n-2n-1}}{3(n-2)} t_{n-1}} \\ &= \frac{f_{n-2}(t_{n-3}, t_{n-1})}{t_{n-1}}.\end{aligned}\quad (3.66)$$

Then, from (3.64)–(3.66), we infer that

$$\begin{aligned}\frac{dg_{n-1}(t_n)}{dt_n} &= \frac{\frac{\partial f_{n-1}(t_{n-2}, t_n)}{\partial t_n}}{1 - \frac{\frac{\partial f_{n-1}(t_{n-2}, t_n)}{\partial t_{n-2}} \frac{\partial t_{n-2}}{\partial t_{n-1}}}} \\ &< \frac{\frac{\frac{na_{n-1n}}{4(n-1)} f_{n-1}(t_{n-2}, t_n)}{\frac{3(n-2)a_{n-1n-2}}{4(n-1)} t_{n-2} + \frac{na_{n-1n}}{4(n-1)} t_n}}{1 - \frac{\frac{3(n-2)a_{n-1n-2}}{4(n-1)} f_{n-1}(t_{n-2}, t_n)}{\frac{3(n-2)a_{n-1n-2}}{4(n-1)} t_{n-2} + \frac{na_{n-1n}}{4(n-1)} t_n} \frac{f_{n-2}(t_{n-3}, t_{n-1})}{t_{n-1}}} \\ &= \frac{f_{n-1}(t_{n-2}, t_n)}{t_n} = \frac{t_{n-1}}{t_n} = \frac{g_{n-1}(t_n)}{t_n}.\end{aligned}\quad (3.67)$$

Hence, using (3.67) and the constraints (3.35), we have

$$\begin{aligned}\frac{dF(t_n)}{dt_n} &= 1 - \frac{\partial f_n(t_{n-1})}{\partial t_{n-1}} \frac{dt_{n-1}}{dt_n} \Big|_{t_{n-1}=g_{n-1}(t_n)} \\ &= 1 - \frac{\frac{(n-1)a_{nn-1}}{n} f_n(g_{n-1}(t_n))}{\sqrt{P_n^2(\mathbf{w}; g_{n-1}(t_n)) - \frac{4b_n a_{nn}}{n\lambda}}} \frac{dg_{n-1}(t_n)}{dt_n} \\ &> 1 - \frac{\frac{(n-1)a_{nn-1}}{n} f_n(g_{n-1}(t_n))}{\frac{(n-1)a_{nn-1}}{n} g_{n-1}(t_n)} \frac{g_{n-1}(t_n)}{t_n} \\ &= 1 - \frac{f_n(g_{n-1}(t_n))}{t_n} = \frac{F(t_n)}{t_n}.\end{aligned}\quad (3.68)$$

Therefore, we conclude from (3.68) that  $F(\cdot)$  is strictly increasing over  $(0, \infty)$ , which implies the uniqueness of the zero of  $F(\cdot)$ . Then the proof of Proposition 3.1 is complete.  $\square$

### 3.3 Constrained minimization

By Proposition 3.1, for any  $\mathbf{w} \in \mathcal{A}$ , we see that the equations (3.23)–(3.25) with respect to  $(c_1, \dots, c_n)$  admit a solution  $(c_1(\mathbf{w}), \dots, c_n(\mathbf{w}))$  given by (3.38)–(3.40), such that  $\mathbf{v}$  defined by

$$v_i = w_i + c_i(\mathbf{w}), \quad i = 1, \dots, n, \quad (3.69)$$

satisfies (3.6).

Therefore, to find the critical points of the functional  $I$ , we consider the functional

$$J(\mathbf{w}) \equiv I(w_1 + c_1(\mathbf{w}), \dots, w_n + c_n(\mathbf{w})) = I(\mathbf{w} + \mathbf{c}(\mathbf{w})), \quad (3.70)$$

where  $\mathbf{w} \in \mathcal{A}$ .

Noting that (3.6) is equivalent to (3.22), and multiplying (3.22) by  $\mathbf{1}^\tau$ , we see that

$$\int_{\Omega} \mathbf{U}^\tau \tilde{S}(\mathbf{U} - \mathbf{1}) dx + \frac{\mathbf{1}^\tau \mathbf{b}}{\lambda} = 0. \quad (3.71)$$

Then in view of  $\tilde{K}^{-1} \mathbf{1} = \mathbf{1}$ , that is  $\tilde{S}^{-1} \tilde{P}^{-1} \mathbf{1} = \mathbf{1}$ , we have

$$- \int_{\Omega} \mathbf{1}^\tau \tilde{P}^{-1} (\mathbf{1} - \mathbf{U}) dx + \int_{\Omega} (\mathbf{U} - \mathbf{1})^\tau \tilde{S}(\mathbf{U} - \mathbf{1}) dx + \frac{\mathbf{1}^\tau \mathbf{b}}{\lambda} = 0. \quad (3.72)$$

Hence from (3.72) we may write the functional  $J$  as

$$J(\mathbf{w}) = \frac{1}{2} \sum_{i=1}^2 \int_{\Omega} \partial_i \mathbf{w}^\tau M \partial_i \mathbf{w} dx + \frac{\lambda}{2} \int_{\Omega} \mathbf{1}^\tau \tilde{P}^{-1} (\mathbf{1} - \mathbf{U}) dx + \mathbf{b}^\tau \mathbf{c} - \frac{\mathbf{1}^\tau \mathbf{b}}{2}, \quad (3.73)$$

which is

$$\begin{aligned} J(\mathbf{w}) &= \frac{1}{2} \sum_{i=1}^2 \int_{\Omega} \partial_i \mathbf{w}^\tau M \partial_i \mathbf{w} dx + \lambda \sum_{i=1}^n \frac{i(n+1-i)}{2n} \int_{\Omega} \left(1 - e^{c_i} e^{u_i^0 + w_i}\right) dx \\ &\quad + \sum_{i=1}^n b_i c_i - \frac{1}{2} \sum_{i=1}^n b_i. \end{aligned} \quad (3.74)$$

It is easy to see that the functional  $J$  is Frechét differentiable in the interior of  $\mathcal{A}$ . If we find a minimizer  $\mathbf{w}$  of  $J$ , which lies in the interior of  $\mathcal{A}$ , then  $(\mathbf{w} + \mathbf{c}(\mathbf{w}))$  is a critical point of  $I$ . Hence, we just need to find a minimizer of  $J$  in the interior of  $\mathcal{A}$ , denoted by  $\text{int}\mathcal{A}$ .

Below we first aim to find a minimizer of  $J$  in  $\mathcal{A}$ . We begin by establishing the following lemma.

**Lemma 3.1** For any  $\mathbf{w} \in \mathcal{A}$ , there hold

$$e^{c_i} \int_{\Omega} e^{u_i^0 + w_i} dx \leq |\Omega|, \quad i = 1, \dots, n, \quad (3.75)$$

$$e^{c_i} \leq 1, \quad i = 1, \dots, n. \quad (3.76)$$

**Proof.** Using (3.38)–(3.40) and the constraints (3.36), we have

$$e^{c_1} \leq \frac{1}{a_{11}} \left( \frac{a_1}{n} + \frac{(n-1)a_{12}}{n} e^{c_2} \right), \quad (3.77)$$

$$e^{c_i} \leq \frac{1}{a_{ii}} \left( \frac{a_i}{i(n+1-i)} + \frac{(i-1)(n+2-i)a_{ii-1}}{2i(n+1-i)} e^{c_{i-1}} \right. \\ \left. + \frac{(i+1)(n-i)a_{ii+1}}{2i(n+1-i)} e^{c_{i+1}} \right), \quad i = 2, \dots, n-1, \quad (3.78)$$

$$e^{c_n} \leq \frac{1}{a_{nn}} \left( \frac{a_n}{n} + \frac{(n-1)a_{nn-1}}{n} e^{c_{n-1}} \right). \quad (3.79)$$

Let us define an  $n \times n$  tridiagonal matrix  $A$  by

$$A \equiv \begin{pmatrix} na_{11} & -(n-1)a_{12} & 0 & \cdots & 0 \\ -\frac{n}{2}a_{21} & 2(n-1)a_{22} & -\frac{3(n-2)}{2}a_{23} & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & -A_{ii-1} & A_{ii} & -A_{ii+1} & \cdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -\frac{3(n-2)}{2}a_{n-1n-2} & 2(n-1)a_{n-1n-1} & -\frac{n}{2}a_{n-1n} \\ 0 & \cdots & 0 & -(n-1)a_{nn-1} & na_{nn} \end{pmatrix}, \quad (3.80)$$

where

$$A_{ii-1} \equiv \frac{(i-1)(n+2-i)}{2} a_{ii-1}, \quad A_{ii} \equiv i(n+1-i)a_{ii}, \quad (3.81)$$

$$A_{ii+1} \equiv \frac{(i+1)(n-i)}{2} a_{ii+1}, \quad i = 3, \dots, n-2, \quad (3.82)$$

and we use the notation (3.30).

We use the convention that for any two vectors  $\mathbf{a} = (a_1, \dots, a_n)^\tau$  and  $\mathbf{b} = (b_1, \dots, b_n)^\tau$  we write  $\mathbf{a} \leq \mathbf{b}$  if  $a_i \leq b_i$  for all  $i = 1, \dots, n$ .

Thus the inequalities (3.77)–(3.79) can be rewritten as

$$A(e^{c_1}, \dots, e^{c_n})^\tau \leq (a_1, \dots, a_n)^\tau. \quad (3.83)$$

Now we denote the adjugate matrices of  $A$  and  $\tilde{K}$  by

$$A^* = (A_{ij}^*)_{n \times n}, \quad (3.84)$$

$$\tilde{K}^* = (\tilde{K}_{ij}^*)_{n \times n}, \quad (3.85)$$

respectively.

By using Hölder's inequality and an induction argument we see that all the entries of  $A^*$  and the determinant of  $A$  are positive. Then we may express  $A^{-1}$  as

$$A^{-1} = \frac{1}{\det A} A^*, \quad (3.86)$$

whose entries are all positive. Hence from (3.83) we obtain

$$\begin{pmatrix} e^{c_1} \\ \vdots \\ e^{c_i} \\ \vdots \\ e^{c_n} \end{pmatrix} \leq A^{-1} \begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_n \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sum_{j=1}^n a_j A_{1j}^*}{na_{11}A_{11}^* - (n-1)a_{12}A_{21}^*} \\ \vdots \\ \frac{\sum_{j=1}^n a_j A_{ij}^*}{-\frac{(i-1)(n+2-i)}{2}a_{ii-1}A_{i-1i}^* + i(n+1-i)a_{ii}A_{ii}^* - \frac{(i+1)(n-i)}{2}a_{i+1i}A_{i+1i}^*} \\ \vdots \\ \frac{\sum_{j=1}^n a_j A_{nj}^*}{-(n-1)a_{nn-1}A_{n-1n}^* + na_{nn}A_{nn}^*} \end{pmatrix}. \quad (3.87)$$



Therefore, by repeatedly using Hölder's inequality, we have

$$\begin{aligned}
\begin{pmatrix} e^{c_1} a_1 \\ \vdots \\ e^{c_i} a_i \\ \vdots \\ e^{c_n} a_n \end{pmatrix} &\leq \begin{pmatrix} \frac{\sum_{j=1}^n a_1 a_j A_{1j}^*}{na_{11}A_{11}^* - (n-1)a_{12}A_{21}^*} \\ \vdots \\ \frac{\sum_{j=1}^n a_i a_j A_{ij}^*}{-\frac{(i-1)(n+2-i)}{2}a_{ii-1}A_{i-1i}^* + i(n+1-i)a_{ii}A_{ii}^* - \frac{(i+1)(n-i)}{2}a_{ii+1}A_{i+1i}^*} \\ \vdots \\ \frac{\sum_{j=1}^n a_n a_j A_{nj}^*}{-(n-1)a_{nn-1}A_{n-1n}^* + na_{nn}A_{nn}^*} \end{pmatrix} \\
&\leq \begin{pmatrix} \frac{|\Omega| \prod_{j=1}^n a_{jj} \sum_{j=1}^n \tilde{K}_{1j}^*}{\prod_{j=1}^n a_{jj} \det \tilde{K}} \\ \vdots \\ \frac{|\Omega| \prod_{j=1}^n a_{jj} \sum_{j=1}^n \tilde{K}_{ij}^*}{\prod_{j=1}^n a_{jj} \det \tilde{K}} \\ \vdots \\ \frac{|\Omega| \prod_{j=1}^n a_{jj} \sum_{j=1}^n \tilde{K}_{nj}^*}{\prod_{j=1}^n a_{jj} \det \tilde{K}} \end{pmatrix} = |\Omega| \begin{pmatrix} \sum_{j=1}^n (\tilde{K}^{-1})_{1j} \\ \vdots \\ \sum_{j=1}^n (\tilde{K}^{-1})_{ij} \\ \vdots \\ \sum_{j=1}^n (\tilde{K}^{-1})_{nj} \end{pmatrix} = |\Omega| \tilde{K}^{-1} \mathbf{1} = |\Omega| \mathbf{1}. \quad (3.88)
\end{aligned}$$

Then the proof of Lemma 3.1 is complete.  $\square$

**Lemma 3.2** *For any  $\mathbf{w} \in \mathcal{A}$  and  $s \in (0, 1)$ , there holds*

$$\int_{\Omega} e^{u_i^0 + w_i} dx \leq \left( \frac{\lambda}{4nb_i} \right)^{\frac{1-s}{s}} \left( \int_{\Omega} e^{su_i^0 + sw_i} dx \right)^{\frac{1}{s}}, \quad i = 1, \dots, n. \quad (3.89)$$

For a proof of this lemma, see [43, 44].

To proceed further, we need the well-known Moser–Trudinger inequality (see [2, 17])

$$\int_{\Omega} e^w dx \leq C \exp \left( \frac{1}{16\pi} \|\nabla w\|_2^2 \right), \quad \forall w \in \dot{W}^{1,2}(\Omega), \quad (3.90)$$

where  $C$  is a positive constant depending on  $\Omega$  only.

Noting that the matrices  $M$  and  $\tilde{S}$ , defined by (3.12) and (3.15), are both positive definite, we have the following coercive estimate for  $J$ .

**Lemma 3.3** *For any  $\mathbf{w} \in \mathcal{A}$  there exists a positive constant  $C$  independent of  $\lambda$  such that*

$$J(\mathbf{w}) \geq \frac{\alpha_0}{4} \sum_{i=1}^n \|\nabla w_i\|_2^2 - C(\ln \lambda + 1), \quad (3.91)$$

where  $\alpha_0 > 0$  is the smallest eigenvalue of  $M$ .

**Proof.** Since the matrices  $M$  and  $\tilde{S}$ , defined by (3.12) and (3.15), are both positive definite, denoting by  $\alpha_0$  the smallest eigenvalue of  $M$ , we have

$$J(\mathbf{w}) \geq \frac{\alpha_0}{2} \sum_{i=1}^n \|\nabla w_i\|_2^2 + \sum_{i=1}^n b_i c_i. \quad (3.92)$$

Using (3.38)–(3.40), we have

$$e^{c_i} \geq \frac{a_i}{2i(n+1-i)a_{ii}} = \frac{\int_{\Omega} e^{u_i^0 + w_i} dx}{2i(n+1-i) \int_{\Omega} e^{2u_i^0 + 2w_i} dx}. \quad (3.93)$$

Then by the constraints (3.36) we obtain

$$e^{c_i} \geq \frac{2nb_i}{i(n+1-i)\lambda \int_{\Omega} e^{u_i^0 + w_i} dx}, \quad (3.94)$$

which implies

$$c_i \geq \ln \frac{2nb_i}{i(n+1-i)} - \ln \lambda - \ln \int_{\Omega} e^{u_i^0 + w_i} dx, \quad i = 1, \dots, n. \quad (3.95)$$

In view of Lemma 3.2, and the Moser–Trudinger inequality, we estimate the last term in (3.95) with

$$\begin{aligned} & \ln \int_{\Omega} e^{u_i^0 + w_i} dx \\ & \leq \frac{1}{s} \ln \int_{\Omega} e^{su_i^0 + sw_i} dx + \frac{1-s}{s} (\ln \lambda - \ln 4nb_i) \\ & \leq \frac{1}{s} \left( \ln C + s \max_{\Omega} u_i^0 + \frac{s^2}{16\pi} \|\nabla w_i\|_2^2 \right) + \frac{1-s}{s} (\ln \lambda - \ln 4nb_i) \\ & \leq \frac{s}{16\pi} \|\nabla w_i\|_2^2 + \frac{1-s}{s} (\ln \lambda - \ln 4nb_i) + \frac{1}{s} \ln C + \max_{\Omega} u_i^0, \quad i = 1, \dots, n. \end{aligned} \quad (3.96)$$

Then inserting (3.96) into (3.95) gives

$$\begin{aligned} c_i & \geq -\frac{s}{16\pi} \|\nabla w_i\|_2^2 - \frac{1}{s} (\ln \lambda + \ln C - \ln 4nb_i) \\ & \quad - \ln 2i(n+1-i) - \max_{\Omega} u_i^0, \quad i = 1, \dots, n. \end{aligned} \quad (3.97)$$

Hence combining (3.92) and (3.97) we get

$$\begin{aligned} J(\mathbf{w}) & \geq \left( \frac{\alpha_0}{2} - \frac{s}{16\pi} \right) \sum_{i=1}^n \|\nabla w_i\|_2^2 - \frac{1}{s} \sum_{i=1}^n b_i (\ln \lambda + \ln C - \ln 4nb_i) \\ & \quad - \sum_{i=1}^n b_i \left( \ln 2i[n+1-i] + \max_{\Omega} u_i^0 \right), \end{aligned} \quad (3.98)$$

which concludes the lemma with taking  $s$  suitably small.  $\square$

Noting that  $J$  is weakly lower semicontinuous in  $\mathcal{A}$  and using Lemma 3.3, we infer that  $J$  has a minimizer in  $\mathcal{A}$ .

### 3.4 Interior minimizer

In the sequel we show that the minimizer of  $J$  obtained above is an interior point of  $\mathcal{A}$  when  $\lambda$  is suitably large. To this end, we first estimate the value of the functional  $J$  on the boundary of  $\mathcal{A}$ .

**Lemma 3.4** *On the boundary of  $\mathcal{A}$  there exists a constant  $C > 0$  independent of  $\lambda$  such that*

$$\inf_{\mathbf{w} \in \partial \mathcal{A}} J(\mathbf{w}) \geq \frac{|\Omega|\lambda}{2} - C(\ln \lambda + \sqrt{\lambda} + 1). \quad (3.99)$$

**Proof.** On the boundary of  $\mathcal{A}$ , at least one of the following  $n$  conditions occurs:

$$a_i^2 = \frac{4nb_i a_{ii}}{\lambda}, \quad i = 1, \dots, n. \quad (3.100)$$

Without loss of generality, if  $i = 1$  in (3.100), then using (3.88) and Hölder's inequality, we conclude

$$\begin{aligned} e^{c_1} a_1 &\leq \frac{\sum_{j=1}^n a_1 a_j A_{1j}^*}{na_{11} A_{11}^* - (n-1)a_{12} A_{21}^*} \\ &\leq \frac{a_1^2 \prod_{j=2}^n a_{jj} \tilde{K}_{11}^*}{a_{11} \prod_{j=2}^n a_{jj} \det \tilde{K}} + \frac{a_1 \sqrt{|\Omega|} \prod_{j=2}^n a_{jj} \sum_{j=2}^n \tilde{K}_{1j}^*}{\sqrt{a_{11}} \prod_{j=2}^n a_{jj} \det \tilde{K}} \\ &= (\tilde{K}^{-1})_{11} \frac{a_1^2}{a_{11}} + \sqrt{|\Omega|} \frac{a_1}{\sqrt{a_{11}}} \sum_{j=2}^n (\tilde{K}^{-1})_{1j} \\ &= \frac{8nb_1}{(n+1)\lambda} + \frac{2(n-1)\sqrt{nb_1|\Omega|}}{(n+1)\sqrt{\lambda}}. \end{aligned} \quad (3.101)$$

Hence using Lemma 3.1 and (3.101), we infer that

$$\begin{aligned} &\lambda \sum_{i=1}^n \frac{i(n+1-i)}{2n} \int_{\Omega} \left(1 - e^{c_i} e^{u_i^0 + w_i}\right) dx \\ &\geq \frac{|\Omega|\lambda}{2} - \frac{1}{n+1} \left(4nb_1 + [n-1]\sqrt{nb_1|\Omega|\lambda}\right). \end{aligned} \quad (3.102)$$

For other cases, we may get similar estimates as (3.102).

Then, estimating  $c_i, i = 1, \dots, n$ , as done in Lemma 3.3, we obtain desired estimate (3.99).  $\square$

At this point, we need to find some suitable test function, which lies in the interior of  $\mathcal{A}$ . We aim to compare the values of the functional at the test function with that on the boundary of  $\mathcal{A}$ .

It was proved in [51] that for  $\mu > 0$  sufficiently large, the problem

$$\Delta v = \mu e^{u_i^0 + v} (e^{u_i^0 + v} - 1) + \frac{4\pi N_i}{|\Omega|} \quad \text{in } \Omega, \quad i = 1, \dots, n, \quad (3.103)$$

admits a solution  $v_i^\mu$ , satisfying  $u_i^0 + v_i^\mu < 0$  in  $\Omega$ ,  $c_i^\mu = \frac{1}{|\Omega|} \int_\Omega v_i^\mu dx \rightarrow 0$ , and  $w_i^\mu = v_i^\mu - c_i^\mu \rightarrow -u_i^0$  pointwise as  $\mu \rightarrow \infty$ ,  $i = 1, \dots, n$ . In particular, we have the limits

$$\lim_{\mu \rightarrow \infty} \int_\Omega e^{u_i^0 + w_i^\mu} dx = |\Omega|, \quad \lim_{\mu \rightarrow \infty} \int_\Omega e^{2u_i^0 + 2w_i^\mu} dx = |\Omega|, \quad i = 1, \dots, n. \quad (3.104)$$

Let us introduce an  $n \times n$  tridiagonal matrix  $\tilde{A}(\mathbf{w}^\mu)$  defined as

$$\tilde{A}(\mathbf{w}^\mu) \equiv \begin{pmatrix} na_{11} & -(n-1)|\Omega| & 0 & \dots & 0 \\ -\frac{n}{2}|\Omega| & 2(n-1)a_{22} & -\frac{3(n-2)}{2}|\Omega| & \dots & 0 \\ \ddots & \ddots & \ddots & \ddots & \vdots \\ \dots & -\frac{(i-1)(n+2-i)}{2}|\Omega| & i(n+1-i)a_{ii} & -\frac{(i+1)(n-i)}{2}|\Omega| & \dots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & -\frac{3(n-2)}{2}|\Omega| & 2(n-1)a_{n-1n-1} & -\frac{n}{2}|\Omega| \\ 0 & \dots & 0 & -(n-1)|\Omega| & na_{nn} \end{pmatrix}, \quad (3.105)$$

where we use the notation (3.30) with  $a_{ii} = a_{ii}(w_i^\mu)$ ,  $i = 1, \dots, n$ . Then we see from (3.104) that

$$\lim_{\mu \rightarrow \infty} \tilde{A}(\mathbf{w}^\mu) = |\Omega| \tilde{K}. \quad (3.106)$$

It is easy to see from Jensen's inequality that all the cofactors and the determinant of  $\tilde{A}(\mathbf{w}^\mu)$  are positive, which implies in particular  $\tilde{A}(\mathbf{w}^\mu)$  is invertible and all the entries of  $\tilde{A}^{-1}(\mathbf{w}^\mu)$  are positive. Then it follows from (3.106) that

$$\lim_{\mu \rightarrow \infty} \tilde{A}^{-1}(\mathbf{w}^\mu) = \frac{1}{|\Omega|} \tilde{K}^{-1}. \quad (3.107)$$

Hence we conclude from the limits (3.104), (3.107), and the definition of  $\mathcal{A}$  that, for a fixed  $\tilde{\lambda}_0 > 0$  large and for any  $\varepsilon \in (0, 1)$ , there exists a  $\mu_\varepsilon \gg 1$ , such that

$$\mathbf{w}^{\mu_\varepsilon} = (w_1^{\mu_\varepsilon}, \dots, w_n^{\mu_\varepsilon}) \in \text{int} \mathcal{A}, \quad (3.108)$$

for every  $\lambda > \tilde{\lambda}_0$ , and there hold

$$\text{diag} \left\{ a_{11}(w_1^{\mu_\varepsilon}), \dots, a_{nn}(w_n^{\mu_\varepsilon}) \right\} < 2|\Omega| \text{diag} \left\{ 1, \dots, 1 \right\}, \quad (3.109)$$

$$\frac{(1-\varepsilon)}{|\Omega|} \tilde{K}^{-1} \leq \tilde{A}^{-1}(\mathbf{w}^{\mu_\varepsilon}) \leq \frac{(1+\varepsilon)}{|\Omega|} \tilde{K}^{-1} < \frac{2}{|\Omega|} \tilde{K}^{-1}. \quad (3.110)$$

Here we use the convention that for  $n \times n$  matrices  $G = (g_{ij})_{n \times n}$  and  $H = (h_{ij})_{n \times n}$ , we write  $G \leq H$  if  $g_{ij} \leq h_{ij}$  for all  $i, j = 1, \dots, n$ .

Now we prove the following comparison result.

**Lemma 3.5** When  $\lambda > 0$  is suitably large, for the test vector  $\mathbf{w}^{\mu_\varepsilon}$  given by (3.108), there holds

$$J(\mathbf{w}^{\mu_\varepsilon}) - \inf_{\mathbf{w} \in \partial A} J(\mathbf{w}) < -1. \quad (3.111)$$

**Proof.** For  $\mathbf{w}^{\mu_\varepsilon}$  given by (3.108), using Proposition 3.1, we may get a corresponding vector  $(c_1(\mathbf{w}^{\mu_\varepsilon}), \dots, c_n(\mathbf{w}^{\mu_\varepsilon}))$  defined by (3.38)–(3.40). Then applying Jensen's inequality and (3.38)–(3.40), we obtain

$$\begin{aligned} e^{c_1(\mathbf{w}^{\mu_\varepsilon})} &= \frac{P_1(\mathbf{w}^{\mu_\varepsilon}; e^{c_2(\mathbf{w}^{\mu_\varepsilon})})}{2a_{11}(w_1^{\mu_\varepsilon})} \left( 1 + \sqrt{1 - \frac{4b_1a_{11}(w_1^{\mu_\varepsilon})}{n\lambda P_1^2(\mathbf{w}^{\mu_\varepsilon}; e^{c_2(\mathbf{w}^{\mu_\varepsilon})})}} \right) \\ &\geq \frac{P_1(\mathbf{w}^{\mu_\varepsilon}; e^{c_2(\mathbf{w}^{\mu_\varepsilon})})}{a_{11}(w_1^{\mu_\varepsilon})} - \frac{2b_1}{n\lambda P_1(\mathbf{w}^{\mu_\varepsilon}; e^{c_2(\mathbf{w}^{\mu_\varepsilon})})} \\ &\geq \frac{|\Omega|}{a_{11}(w_1^{\mu_\varepsilon})} \left( \frac{1}{n} + \frac{n-1}{n} e^{c_2(\mathbf{w}^{\mu_\varepsilon})} \right) - \frac{2b_1}{\lambda|\Omega|}. \end{aligned} \quad (3.112)$$

Analogously,

$$\begin{aligned} e^{c_i(\mathbf{w}^{\mu_\varepsilon})} &\geq \frac{|\Omega|}{a_{ii}(w_i^{\mu_\varepsilon})} \left\{ \frac{1}{i(n+1-i)} + \frac{(i-1)(n+2-i)}{2i(n+1-i)} e^{c_{i-1}(\mathbf{w}^{\mu_\varepsilon})} \right. \\ &\quad \left. + \frac{(i+1)(n-i)}{2i(n+1-i)} e^{c_{i+1}(\mathbf{w}^{\mu_\varepsilon})} \right\} - \frac{2nb_i}{\lambda i(n+1-i)|\Omega|}, \quad i = 2, \dots, n-1, \end{aligned} \quad (3.113)$$

$$e^{c_n(\mathbf{w}^{\mu_\varepsilon})} \geq \frac{|\Omega|}{a_{nn}(w_n^{\mu_\varepsilon})} \left( \frac{1}{n} + \frac{n-1}{n} e^{c_{n-1}(\mathbf{w}^{\mu_\varepsilon})} \right) - \frac{2b_n}{\lambda|\Omega|}, \quad (3.114)$$

where we use the notation (3.30) with the understanding that  $a_{ii} = a_{ii}(w_i^{\mu_\varepsilon})$ ,  $i = 1, \dots, n$ .

Noting the definition (3.105), we may express the inequalities (3.112)–(3.114) equivalently as

$$\tilde{A}(\mathbf{w}^{\mu_\varepsilon}) (e^{c_1(\mathbf{w}^{\mu_\varepsilon})}, \dots, e^{c_n(\mathbf{w}^{\mu_\varepsilon})})^\tau \geq |\Omega| \mathbf{1} - \frac{2n}{\lambda|\Omega|} \text{diag}\{a_{11}(w_1^{\mu_\varepsilon}), \dots, a_{nn}(w_n^{\mu_\varepsilon})\} \mathbf{b}. \quad (3.115)$$

Since all the entries of  $\tilde{A}^{-1}(\mathbf{w}^{\mu_\varepsilon})$  are positive, we infer from (3.115), (3.109), and (3.110) that

$$\begin{aligned} &(e^{c_1(\mathbf{w}^{\mu_\varepsilon})}, \dots, e^{c_n(\mathbf{w}^{\mu_\varepsilon})})^\tau \\ &\geq \tilde{A}^{-1}(\mathbf{w}^{\mu_\varepsilon}) \left( |\Omega| \mathbf{1} - \frac{2n}{\lambda|\Omega|} \text{diag}\{a_{11}(w_1^{\mu_\varepsilon}), \dots, a_{nn}(w_n^{\mu_\varepsilon})\} \mathbf{b} \right) \\ &\geq \tilde{A}^{-1}(\mathbf{w}^{\mu_\varepsilon}) \left( |\Omega| \mathbf{1} - \frac{4n}{\lambda} \mathbf{b} \right) \\ &\geq (1 - \varepsilon) \tilde{K}^{-1} \mathbf{1} - \frac{8n}{\lambda|\Omega|} \tilde{K}^{-1} \mathbf{b} \\ &= (1 - \varepsilon) \mathbf{1} - \frac{8n}{\lambda|\Omega|} R^{-1} K^{-1} \mathbf{b}. \end{aligned} \quad (3.116)$$

Then, it follows from (3.116) that

$$\int_{\Omega} \left(1 - e^{c_i(\mathbf{w}^{\mu_\varepsilon})} e^{u_i^0 + w_i^{\mu_\varepsilon}}\right) dx \leq |\Omega|\varepsilon + \frac{16n}{\lambda i(n+1-i)} \sum_{j=1}^n (K^{-1})_{ij} b_j, \quad i = 1, \dots, n. \quad (3.117)$$

At this point using (3.75) and (3.117) we conclude that, for any small  $\varepsilon > 0$ , there exists a positive constant  $C_\varepsilon$  independent of  $\lambda$  such that

$$J(\mathbf{w}^{\mu_\varepsilon}) \leq \frac{(n+1)(n+2)|\Omega|\lambda}{12}\varepsilon + C_\varepsilon. \quad (3.118)$$

Consequently, from (3.118) and Lemma 3.5, we infer that

$$J(\mathbf{w}^{\mu_\varepsilon}) - \inf_{\mathbf{w} \in \partial\mathcal{A}} J(\mathbf{w}) \leq \frac{|\Omega|\lambda}{2} \left( \frac{[n+1][n+2]}{6}\varepsilon - 1 \right) + C(\ln \lambda + \sqrt{\lambda} + 1), \quad (3.119)$$

where  $C > 0$  is a constant independent of  $\lambda$ . Hence, by taking  $\varepsilon$  suitably small and  $\lambda$  sufficiently large in (3.119), we obtain the desired estimate (3.111).  $\square$

Now we may infer from Lemma 3.3 and 3.5 that there exists a  $\lambda_1 > 0$  such that, for every  $\lambda > \lambda_1$ , the functional  $J$  admits a minimizer

$$\mathbf{w}^\lambda \equiv (w_1^\lambda, \dots, w_n^\lambda)^\tau \in \text{int}\mathcal{A}. \quad (3.120)$$

### 3.5 Solution to the original system

Since we use a constrained minimization to get a minimizer  $\mathbf{w}^\lambda$  of  $J(\mathbf{w}) = I(\mathbf{w} + \mathbf{c}(\mathbf{w}))$  in the subspace of  $W^{1,2}(\Omega)$ , it is not obvious that whether  $\mathbf{w}^\lambda$  gives rise to a solution of the system (3.14). Here we show that  $\mathbf{v}^\lambda \equiv (v_1^\lambda, \dots, v_n^\lambda)$  defined by

$$v_i^\lambda = w_i^\lambda + c_i(\mathbf{w}^\lambda), \quad i = 1, \dots, n, \quad (3.121)$$

is actually a solution of the system (3.14).

**Lemma 3.6** *Let  $\mathbf{w}$  be a minimizer of  $J$  in  $\text{int}\mathcal{A}$  and the corresponding vector  $\mathbf{c}(\mathbf{w})$  be determined by (3.38)–(3.40). Then*

$$\mathbf{v} = \mathbf{c}(\mathbf{w}) + \mathbf{w} \quad (3.122)$$

*must be a solution of the system (3.14).*

**Proof.** Since  $\mathbf{w}$  is an interior minimizer of  $J$  in  $\mathcal{A}$ , the Fréchet derivative of  $J(\mathbf{w}) = I(\mathbf{w} + \mathbf{c}(\mathbf{w}))$  at  $\mathbf{w}$  should be zero,

$$[dI(\mathbf{w} + \mathbf{c}(\mathbf{w}))]\mathbf{f} = 0 \quad \text{for any } \mathbf{f} \in \dot{W}^{1,2}(\Omega). \quad (3.123)$$

By the expression of  $I$  (3.18), we rewrite (3.123) in an explicit form

$$\begin{aligned} & \int_{\Omega} \left( \sum_{i=1}^2 \partial_i \mathbf{f}^T M \partial_i \mathbf{w} + \lambda \mathbf{f}^T \mathbf{U} \tilde{S}[\mathbf{U} - \mathbf{1}] \right) dx \\ & + [D_{\mathbf{f}} \mathbf{c}(\mathbf{w})]^T \int_{\Omega} \left( \lambda \mathbf{U} \tilde{S}[\mathbf{U} - \mathbf{1}] + \frac{\mathbf{b}}{|\Omega|} \right) dx = 0 \quad \text{for any } \mathbf{f} \in \dot{W}^{1,2}(\Omega), \end{aligned} \quad (3.124)$$

where

$$D_{\mathbf{f}} \mathbf{c}(\mathbf{w}) = \frac{d}{dt} \mathbf{c}(\mathbf{w} + t\mathbf{f})|_{t=0}, \quad (3.125)$$

is the directional derivative of  $\mathbf{c}$  at  $\mathbf{w}$  along the direction  $\mathbf{f}$ , and the notation (3.5) is used.

Then we use (3.22) to reduce (3.124) into

$$\int_{\Omega} \left( \sum_{i=1}^2 \partial_i \mathbf{f}^T M \partial_i \mathbf{w} + \lambda \mathbf{f}^T \mathbf{U} \tilde{S}[\mathbf{U} - \mathbf{1}] \right) dx = 0. \quad (3.126)$$

Denote by  $L^2(\Omega)$  the scalar-valued or  $n$ -vector-valued function space of  $\Omega$ -period  $L^2$ -functions and decompose  $L^2(\Omega)$  as

$$L^2(\Omega) = \mathbb{R}^n \oplus Y, \quad (3.127)$$

where

$$Y = \left\{ \mathbf{f} \mid \mathbf{f} \in L^2(\Omega), \quad \int_{\Omega} \mathbf{f} dx = \mathbf{0} \right\}. \quad (3.128)$$

We select a vector  $\mathbf{d} \in \mathbb{R}^n$  such that

$$\lambda \mathbf{U} \tilde{S}(\mathbf{U} - \mathbf{1}) + \mathbf{d} \in Y. \quad (3.129)$$

Hence the relation  $\dot{W}^{1,2}(\Omega) \subset Y$  and (3.126) lead to

$$\begin{aligned} 0 &= \int_{\Omega} \left\{ \sum_{i=1}^2 \partial_i \mathbf{f}^T M \partial_i \mathbf{w} + \mathbf{f}^T \left( \lambda \mathbf{U} \tilde{S}[\mathbf{U} - \mathbf{1}] + \mathbf{d} \right) \right\} dx \\ &= \int_{\Omega} \left\{ \sum_{i=1}^2 \partial_i (\mathbf{f} + \mathbf{g})^T M \partial_i \mathbf{w} + (\mathbf{f} + \mathbf{g})^T \left( \lambda \mathbf{U} \tilde{S}[\mathbf{U} - \mathbf{1}] + \mathbf{d} \right) \right\} dx, \end{aligned} \quad (3.130)$$

for any  $\mathbf{g} \in \mathbb{R}^n$ . Consequently, we have

$$\int_{\Omega} \left\{ \sum_{i=1}^2 \partial_i \mathbf{h}^T M \partial_i \mathbf{w} + \mathbf{h}^T \left( \lambda \mathbf{U} \tilde{S}[\mathbf{U} - \mathbf{1}] + \mathbf{d} \right) \right\} dx = 0 \quad \text{for any } \mathbf{h} \in W^{1,2}(\Omega). \quad (3.131)$$

Then we conclude from (3.131) that  $\mathbf{w}$  is a smooth solution of the system

$$\Delta M \mathbf{w} = \lambda \mathbf{U} \tilde{S}(\mathbf{U} - \mathbf{1}) + \mathbf{d}, \quad (3.132)$$

which, after being integrated over  $\Omega$ , gives us

$$\lambda \int_{\Omega} \mathbf{U} \tilde{S}(\mathbf{U} - \mathbf{1}) dx + \mathbf{d}|\Omega| = \mathbf{0}. \quad (3.133)$$

Hence we infer from (3.133) and (3.22) that

$$\mathbf{d} = \frac{\mathbf{b}}{|\Omega|}. \quad (3.134)$$

Combining (3.132) and (3.134) we see that  $\mathbf{v} = \mathbf{c}(\mathbf{w}) + \mathbf{w}$  is a solution of (3.14). Thus the proof of Lemma 3.6 is complete.  $\square$

At this stage we infer from Lemma 3.6 that when  $\lambda > \lambda_1$ ,  $\mathbf{v}^\lambda$  defined by (3.121) is a solution of (3.14). Therefore part (ii) of Theorem 2.1 follows.

### 3.6 Asymptotic behavior and quantized integrals

In this subsection we study the asymptotic behavior of the solution  $\mathbf{v}^\lambda$  of (3.14) defined by (3.121) when  $\lambda \rightarrow \infty$  and establish the quantized integrals as stated in Theorem 2.1.

**Lemma 3.7** *Let  $\mathbf{v}^\lambda$  be the solution of (3.14) given by (3.121). Then there holds*

$$\lim_{\lambda \rightarrow \infty} \int_{\Omega} \left( e^{u_i^0 + v_i^\lambda} - 1 \right)^2 dx = 0, \quad i = 1, \dots, n. \quad (3.135)$$

**Proof.** Since  $J$  achieves its minimum at  $\mathbf{w}^\lambda \in \text{int}\mathcal{A}$ , we see from (3.118) that, for any  $\varepsilon \in (0, 1)$ , there exist constants  $\lambda_\varepsilon > 0$  and  $C_\varepsilon > 0$  such that

$$J(\mathbf{w}^\lambda) = \inf_{\mathbf{w} \in \mathcal{A}} J(\mathbf{w}) \leq \frac{(n+1)(n+2)\lambda|\Omega|}{12} \varepsilon + C_\varepsilon \quad \text{for all } \lambda > \lambda_\varepsilon. \quad (3.136)$$

Noting the matrix  $\tilde{S}$  (defined by (3.12)) is positive definite and denoting the smallest eigenvalue of  $\tilde{S}$  by  $\beta_0 > 0$ , we have

$$\int_{\Omega} (\mathbf{U} - \mathbf{1})^\tau \tilde{S}(\mathbf{U} - \mathbf{1}) dx \geq \beta_0 \sum_{i=1}^n \int_{\Omega} \left( e^{u_i^0 + v_i} - 1 \right)^2 dx. \quad (3.137)$$

Therefore, in view of (3.72), (3.74), and (3.137), and estimating  $c_i^\lambda$  as that in Lemma 3.4, we conclude that

$$J(\mathbf{w}^\lambda) \geq \frac{\beta_0 \lambda}{2} \sum_{i=1}^n \int_{\Omega} \left( e^{u_i^0 + v_i^\lambda} - 1 \right)^2 dx - C(\ln \lambda + 1), \quad (3.138)$$

where  $C > 0$  is a constant independent of  $\lambda$ .

Then combining (3.136) and (3.138) leads to

$$\limsup_{\lambda \rightarrow \infty} \sum_{i=1}^n \int_{\Omega} \left( e^{u_i^0 + v_i^\lambda} - 1 \right)^2 dx \leq \frac{(n+1)(n+2)|\Omega|}{6\beta_0} \varepsilon \quad \text{for any } \varepsilon > 0,$$



which implies the desired conclusion (3.135). The proof of Lemma 3.7 is complete.  $\square$

Hence part (iii) of Theorem 2.1 follows from Lemma 3.7.

To establish the quantized integrals (2.14), we just need to integrate the equations (3.3) over  $\Omega$ .

The proof of Theorem 2.1 is complete.

## 4 Concluding remarks

We note that the method to establish Theorem 2.1 can be applied to prove an existence theorem for the problem (2.5) or (2.6) when the matrix  $\tilde{K}$  assumes a more general tridiagonal matrix  $\hat{K}$  form,

$$\hat{K} \equiv \begin{pmatrix} 1 + \alpha_{12} & -\alpha_{12} & \cdots & \cdots & 0 \\ \ddots & \ddots & \ddots & \vdots & \vdots \\ \cdots & -\alpha_{ii-1} & 1 + \alpha_{ii-1} + \alpha_{ii+1} & -\alpha_{ii+1} & \cdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & -\alpha_{nn-1} & 1 + \alpha_{nn-1} \end{pmatrix}, \quad (4.1)$$

where  $\alpha_{12}, \alpha_{ii-1}, \alpha_{ii+1}, \alpha_{nn-1} > 0, i = 2, \dots, n-1$ .

In fact, it is ready to check that all entries of  $\hat{K}^{-1}$  are positive,  $\hat{K}^{-1}$  satisfies  $\hat{K}^{-1}\mathbf{1} = \mathbf{1}$ , and  $\hat{K}$  can be decomposed as

$$\hat{K} = \hat{P}\hat{S}, \quad (4.2)$$

where  $\hat{P}$  is a diagonal matrix with positive diagonal entries and  $\hat{S}$  is a positive definite matrix.

In particular, when

$$\begin{aligned} \alpha_{12} &= n-1 = \alpha_{nn-1}, \\ \alpha_{ii-1} &= \frac{(i-1)(n+2-i)}{2}, \\ \alpha_{ii+1} &= \frac{(i+1)(n-i)}{2}, \quad i = 2, \dots, n-1, \end{aligned} \quad (4.3)$$

the matrix  $\hat{K}$  reduces to  $\tilde{K}$  given by (2.7).

Since the corresponding existence result can be stated in a similar formulation as that of Theorem 2.1, the details are omitted here.

It will be of future interest to develop an existence theory when the Cartan matrix is not tridiagonal.

Han was supported in part by the National Natural Science Foundation of China under grant 11201118 and by the Key Foundation for Henan Colleges under grant 15A110013. Both authors were supported in part by the National Natural Science Foundation of China under grants 11471100 and 11471099.

# References

- [1] A. A. Abrikosov, On the magnetic properties of superconductors of the second group, *Sov. Phys. JETP* **5** (1957) 1174–1182.
- [2] T. Aubin, *Nonlinear Analysis on Manifolds: Monge–Ampère Equations*, Springer, Berlin and New York, 1982.
- [3] H. Brezis and F. Merle, Uniform estimates and blow-up behavior for solutions of  $\Delta u = V(x)e^u$  in two dimensions, *Comm. Partial Diff. Eqns.* **16** (1991) 1223–1254.
- [4] L. Caffarelli and Y. Yang, Vortex condensation in the Chern–Simons Higgs model: an existence theorem, *Commun. Math. Phys.* **168** (1995) 321–336.
- [5] E. Calabi, Isometric imbedding of complex manifolds, *Ann. of Math.* **58** (1953) 1–23.
- [6] S. Y. A. Chang and P. Yang, Prescribing Gaussian curvature on  $S^2$ , *Acta Math.* **159** (1987) 215–259.
- [7] C. C. Chen and C. S. Lin, Sharp estimates for solutions of multi-bubbles in compact Riemann surface, *Comm. Pure Appl. Math.* **55** (2002) 728–771.
- [8] C. C. Chen and C. S. Lin, Topological degree for a mean field equation on Riemann surfaces, *Comm. Pure Appl. Math.* **56** (2003) 1667–1727.
- [9] C. C. Chen and C. S. Lin, Mean field equations of Liouville type with singular data: Sharper estimates, *Discrete Contin. Dyn. Syst.* **28** (2010) 1237–1272.
- [10] W. Chen and C. Li, Classification of solutions of some nonlinear elliptic equations, *Duke Math. J.* **63** (1991) 615–622.
- [11] W. Chen and C. Li, Qualitative properties of solutions to some nonlinear elliptic equations in  $\mathbb{R}^2$ , *Duke Math. J.* **71** (1993) 427–439.
- [12] K. Cheng and W.-M. Ni, On the structure of the conformal Gaussian curvature equation on  $\mathbb{R}^2$ , *Duke Math. J.* **62** (1991) 721–737.
- [13] G. Dunne, Chern–Simons solitons, Toda theories and the chiral model, *Commun. Math. Phys.* **150** (1992) 519–535.
- [14] G. Dunne, *Self-Dual Chern–Simons Theories*, Lecture Notes in Physics, vol. m **36**, Springer, Berlin, 1995.
- [15] G. Dunne, Mass degeneracies in self-dual models, *Phys. Lett. B* **345** (1995) 452–457.
- [16] G. Dunne, R. Jackiw, S.-Y. Pi and C. Trugenberger, Self-dual Chern–Simons solitons and two-dimensional nonlinear equations, *Phys. Rev. D* **43** (1991) 1332–1345.
- [17] L. Fontana, Sharp borderline Sobolev inequalities on compact Riemannian manifolds, *Comment. Math. Helv.* **68** (1993) 415–454.

- [18] J. Fröhlich, The fractional quantum Hall effect, Chern–Simons theory, and integral lattices, *Proc. Internat. Congr. Math.*, pp. 75–105, Birkhäuser, Basel, 1995.
- [19] J. Fröhlich and P. Marchetti, Quantum field theory of anyons, *Lett. Math. Phys.* **16** (1988) 347–358.
- [20] J. Fröhlich and P. Marchetti, Quantum field theory of vortices and anyons, *Commun. Math. Phys.* **121** (1989) 177–223.
- [21] N. Ganoulis, Quantum Toda systems and Lax pairs, *Commun. Math. Phys.* **109** (1987) 23–32.
- [22] N. Ganoulis, P. Goddard, and D. Olive, Self-dual monopoles and Toda molecules, *Nucl. Phys. B* **205** (1982) 601–636.
- [23] X. Han, C. S. Lin, and Y. Yang, Resolution of Chern–Simons–Higgs vortex equations, *Commun. Math. Phys.*, submitted.
- [24] J. Hong, Y. Kim and P.-Y. Pac, Multivortex solutions of the Abelian Chern–Simons–Higgs theory, *Phys. Rev. Lett.* **64** (1990) 2330–2333.
- [25] P. A. Horvathy and P. Zhang, Vortices in (Abelian) Chern–Simons gauge theory, *Phys. Rep.* **481** (2009) 83–142.
- [26] R. Jackiw and E. J. Weinberg, Self-dual Chern–Simons vortices, *Phys. Rev. Lett.* **64** (1990) 2334–2337.
- [27] J. Jost, C. S. Lin, and G. Wang, Analytic aspects of Toda system: II. Bubbling behavior and existence of solutions, *Comm. Pure Appl. Math.* **59** (2006) 526–558.
- [28] J. Jost and G. Wang, Classification of solutions of a Toda system in  $\mathbb{R}^2$ , *Int. Math. Res. Not.* **6** (2002) 277–290.
- [29] J. L. Kazdan and F. W. Warner, Curvature functions for compact 2-manifolds, *Ann. of Math.* **99** (1974) 14–47.
- [30] J. L. Kazdan and F. W. Warner, Curvature functions for open 2-manifolds, *Ann. of Math.* **99** (1974) 203–219.
- [31] B. Kostant, The solution to a generalized Toda lattice and representation theory, *Adv. Math.* **34** (1979) 195–338.
- [32] A. N. Leznov, On the complete integrability of a nonlinear system of partial differential equations in two-dimensional space, *Theoret. Math. Phys.* **42** (1980) 225–229.
- [33] A. N. Leznov and M. V. Saveliev, Representation of zero curvature for the system of nonlinear partial differential equations  $x_{\alpha, z\bar{z}} = \exp(kx)_{\alpha}$  and its integrability, *Lett. Math. Phys.* **3** (1979) 489–494.
- [34] A. N. Leznov and M. V. Saveliev, Representation theory and integration of nonlinear spherically symmetric equations to gauge theories, *Commun. Math. Phys.* **74** (1980) 111–118.

- [35] Y. Y. Li, Harnack type inequality: the method of moving planes, *Commun. Math. Phys.* **200** (1999) 421–444.
- [36] C. S. Lin, J. Wei, and D. Ye, Classification and nondegeneracy of  $SU(n+1)$  Toda system with singular sources, *Invent. Math.* **190** (2012) 169–207.
- [37] C. S. Lin and L. Zhang, Profile of bubbling solutions to a Liouville system, *Ann. Inst. H. Poincaré – Anal. Non linéaire* **27** (2010) 117–143.
- [38] J. Liouville, Sur l'équation aux différences partielles  $\frac{d^2 \log \lambda}{dudv} \pm \frac{\lambda}{2a^2} = 0$ , *J. de Mathématiques Pures et Appl* **18** (1853) 71–72.
- [39] A. Malchiodi, Topological methods for an elliptic equation with exponential nonlinearities, *Discrete Contin. Dyn. Syst.* **21** (2008) 277–294.
- [40] A. Malchiodi and C.B. Ndiaye, Some existence results for the Toda system on closed surfaces, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* **18** (2007) 391C–412.
- [41] A. Malchiodi and D. Ruiz, A variational analysis of the Toda system on compact surfaces, *Comm. Pure Appl. Math.* **66** (2013) 332–371.
- [42] A. Mikhailov, M. Olshanetsky and A. Perelomov, Two-dimensional generalized Toda lattice, *Commun. Math. Phys.* **79** (1981) 473–488.
- [43] M. Nolasco and G. Tarantello, On a sharp Sobolev-type inequality on two-dimensional compact manifolds. *Arch. Ration. Mech. Anal.* **145** (1998) 161–195.
- [44] M. Nolasco and G. Tarantello, Vortex condensates for the  $SU(3)$  Chern–Simons theory, *Comm. Math. Phys.* **213** (2000) 599–639.
- [45] H. Ohtsuka and T. Suzuki, Blow-up analysis for  $SU(3)$  Toda system, *J. Diff. Eqs.* **232** (2007) 419–440.
- [46] P. Olesen, Soliton condensation in some self-dual Chern–Simons theories, *Phys. Lett. B* **265** (1991) 361–365. Erratum, *B* **267** (1991) 541.
- [47] D. Olive and N. Throk, The symmetries of Dynkin diagrams and the reduction of Toda field equations, *Nucl. Phys. B* **215** (1983) 470–494.
- [48] J. Schiff, Integrability of Chern–Simons–Higgs and Abelian Higgs vortex equations in a background metric, *J. Math. Phys.* **32** (1991) 753–761.
- [49] J. Spruck and Y. Yang, On multivortices in the electroweak theory I: existence of periodic solutions, *Commun. Math. Phys.* **144** (1992) 1–16.
- [50] G. 't Hooft, A property of electric and magnetic flux in nonabelian gauge theories, *Nucl. Phys. B* **153** (1979) 141–160.

- [51] G. Tarantello, Multiple condensate solutions for the Chern–Simons–Higgs theory, *J. Math. Phys.* **37** (1996) 3769–3796.
- [52] S. Wang and Y. Yang, Abrikosov’s vortices in the critical coupling, *SIAM J. Math. Anal.* **23** (1992) 1125–1140.
- [53] E. J. Weinberg and P. Yi., Magnetic monopole dynamics, supersymmetry, and duality, *Phys. Reports* **438** (2007) 65–236.
- [54] F. Wilczek, *Fractional Statistics and Anyonic Superconductivity*, World Scientific, Singapore, 1990.
- [55] Y. Yang, The relativistic non-Abelian Chern–Simons equations, *Commun. Math. Phys.* **186** (1997) 199–218.
- [56] Y. Yang, *Solitons in Field Theory and Nonlinear Analysis*, Springer, New York, 2001.